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An Alternation Characterization of Best Uniform Approximations on Noncompact Intervals*

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We present a characterization for a best uniform approximation to a given bounded continuous function f defined on the real but not necessarily compact interval T from an n-dimensional subspace S of the bounded continuous functions on T. When S is a Haar subspace and each element of S satisfies an additional endpoint regularity condition, such a best approximation may be characterized by an appropriate generalization of the familiar alternation criterion which holds for compact T. One such best approximation that has an alternating error curve may be obtained as the uniform limit of a sequence whose νth term is the unique best uniform approximation to f on the νth member of a suitably chosen expanding sequence of compact subintervals of T. The results apply in the special case where $T = [0, +\infty)$ and S is a family of exponential sums with real exponents.

1. INTRODUCTION

Let $C_b(T)$ denote the space of bounded continuous real valued functions defined on the nondegenerate real interval T with the uniform norm

$$\|f\| = \sup\{|f(t)| : t \in T\}, \tag{1}$$

and let S be an n-dimensional subspace of $C_b(T)$. Well known arguments (cf. 2, p. 20]) show that there is some $y \in S$ which best approximates a given $f \in C_b(T)$ on T with respect to the norm (1). In this paper we formulate a necessary and sufficient condition for y to be such a best approximation. We then specialize this result to the case where S is a Haar subspace (i.e., if $h \in S$ and $\|h\| \neq 0$ then h has at most $n - 1$ distinct zeros in T) which is regular in the sense that for every $h \in S$ the two limits $\lim h(t_l)$, $\lim h(t_r)$ exist as t_l, t_r approach the left, right endpoints of T, respectively.

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In particular, we obtain a simple alternation type characterization which applies in the important special case where $T = [0, \infty)$ and

$$S = \{y : [(D - \lambda_1) \cdots (D - \lambda_n)]\, y = 0\} \tag{2}$$

where $D = d/dt$ is the differential operator and

$$\lambda_1 \leqslant \cdots \leqslant \lambda_{n-1} < 0,\ \lambda_{n-1} \leqslant \lambda_n \leqslant 0. \tag{3}$$

Since the exponents λ_i are real no $h \in S$ with $\| h \| \neq 0$ has more than $n - 1$ distinct zeros (cf. [8, p. 40 #18]), and because of the constraints (3) every $h \in S$ has real limits at $t = 0, +\infty$.

When the interval T is compact and S is a Haar subspace of $C_b(T)$, this alternation type characterization implies the unicity of the best approximation (cf. [2, p. 80]). When the interval of approximation T is not compact, one often loses unicity and the alternation characterization does not apply even when S is a Haar subspace of T. We illustrate this situation (and provide some motivation for the following discussion) by means of the following two examples.

EXAMPLE 1. Let $T = (-1, 1)$, let $u_1(t) = 1 - t^2$, $u_2(t) = t$, and let $f(t) = 1$. When we approximate f using $y = \alpha_1 u_1 + \alpha_2 u_2$ the resulting error function $\epsilon(t) = 1 - \alpha_1(1 - t^2) - \alpha_2 t$ has the minimum sup norm $\| \epsilon \| = 1$ if and only if $\alpha_2 = 0$ and $0 \leqslant \alpha_1 \leqslant 2$. There is no best error curve which alternates at least twice on T although the error curve corresponding to $\alpha_1 = 2$, $\alpha_2 = 0$ does alternate twice on the natural compactification, $[-1, 1]$, of T. ▮

EXAMPLE 2. Let $T = [0, \infty)$, let $u_1(t) = 1$, $u_2(t) = e^{-t}$, and let $f(t) = (1 - e^{-t}) \sin t$. Clearly $y = \alpha_1 u_1 + \alpha_2 u_2$ is a best approximation only if $\alpha_1 = 0$ and $\| f - y \| = 1$ so that $| \alpha_2 | \leqslant 1$, and all such choices of y are best. Although no best approximation alternates even once on T, we see that every optimum error curve ϵ oscillates infinitely often between values arbitrarily close to $\pm \| \epsilon \|$. ▮

The alternation concept can be extended to noncompact intervals as follows.

DEFINITION. We say that $\epsilon \in C_b(T)$ essentially alternates at least n times on T provided that for each $\delta > 0$ there exist $n + 1$ points $t_0 < t_1 < \cdots < t_n$ from T and some $s \in \{-1, +1\}$ such that

$$s \cdot (-1)^i \cdot \epsilon(t_i) \geqslant \| \epsilon \| - \delta, \quad i = 0, 1, \ldots, n.$$

When T is compact and ϵ essentially alternates at least n times on T, then ϵ must also alternate at least n times on T.

2. CHARACTERIZATION

The following theorem provides a general characterization for a best approximation (which may be used even when S is not finite dimensional.)

THEOREM 1. *Let $f \in C_b(T)$, let S be a linear subspace of $C_b(T)$, let $y \in S$, and let $\epsilon = f - y$. Than y is a best approximation to f from S with respect to the norm* (1) *if and only if*

$$\Phi(\epsilon, h) \geqslant 0 \text{ for all } \quad h \in S \tag{4}$$

where

$$\Phi(\epsilon,h) = \lim \sup\{h(t) \operatorname{sgn} \epsilon(t) : t \in T \quad \text{and} \quad | \epsilon(t)| \geqslant \| \epsilon \| - \delta\} \tag{5}$$

with the limit being taken as δ approaches zero through positive values.

Proof. When $\| \epsilon \| > 0$ the same arguments used to establish Lemma 3 in [2] can be used to show that for any $h \in S$ we have

$$\| \epsilon + \alpha h \| = \| \epsilon \| + \alpha\Phi(\epsilon, h) + o(\alpha) \tag{6}$$

as α decreases to zero through positive values. This being the case (4) must hold if y is a best approximation (since (6) shows that if $\Phi(\epsilon, h) < \delta$ for some $h \in S$ then for all sufficiently small $\alpha > 0$ the function $y - \alpha h$ is a better approximation to f than y.)

On the other hand if y fails to be a best approximation, then we can find some $h \in S$ such that $y - h$ is a better approximation so that the constant

$$d = \| \epsilon \| - \| \epsilon + h \|$$

is positive. For $0 < \alpha \leqslant 1$ we have

$$\begin{aligned} \| \epsilon + \alpha h \| &= \|(1 - \alpha) \epsilon + \alpha(\epsilon + h)\| \\ &\leqslant (1 - \alpha) \| \epsilon \| + \alpha \| \epsilon + h \| \\ &= \| \epsilon \| - \alpha d \end{aligned}$$

which when used in conjunction with (6) shows that $\Phi(\epsilon, h) < 0$ so that (4) fails. ∎

Theorem 1 reduces to the real version of Kolomogoroff's characterization [7, p. 15] when T is compact. An alternative characteriation which also applies when T is not compact can be formulated in terms of the dual space of $C_b(T)$, cf. [6, p. 120].

By imposing suitable restrictive hypotheses on f and S we may specialize such general characterizations to forms which are somewhat easier to use in practice. For example, Bram [1] treats the case where S is finite dimensional and where troublesome endpoint conditions are avoided by the assumption that $\{t \in T : |g(t)| \geqslant \delta\}$ is compact whenever $\delta > 0$ and $g \in S \cup \{f\}$. Using Theorem 1 we formulate a simple characterization which applies when S is a finite dimensional Haar space of functions having definite limits at the endpoints of T.

THEOREM 2. *Let S be an n dimensional Haar subspace of $C_b(T)$ and assume that for each $h \in S$ the limits* $\lim h(t_l)$, $\lim h(t_r)$ *exist as t_l, t_r approach the left, right endpoints of T, respectively. Let $f \in C_b(T)$, let $y \in S$, and let $\epsilon = f - y$. Then y is a best approximation to f from S with respect to the norm* (1) *if and only if at least one of the following three conditions holds.*

(i) *The error curve ϵ essentially alternates at least n times on T.*

(ii) *As t approaches some one of the endpoints of T we have* $\limsup |\epsilon(t)| = \|\epsilon\|$ *while* $\lim |h(t)| = 0$ *for every $h \in S$.*

(iii) *As t_l, t_r approach the left, right endpoints of T, respectively,*

$$\limsup\, (-1)^n \cdot \epsilon(t_l) \cdot \epsilon(t_r) = \|\epsilon\|^2$$

while $\lim(-1)^n \cdot h(t_l) \cdot h(t_r) \leqslant 0$ *for every $h \in S$.*

Proof. If (i) holds and $h \in S$ then since S is a Haar subspace we have

$$\sup\{h(t) \operatorname{sgn} \epsilon(t) : t \in T \quad \text{and} \quad |\epsilon(t)| \geqslant \|\epsilon\| - \delta\} \geqslant 0$$

whenever $\delta > 0$ is sufficiently small, and in the limit $\delta \to 0+$ we find that $\Phi(\epsilon, h) \geqslant 0$, i.e., (4) holds. Likewise (ii), (iii) each imply (4). Together with Theorem 1 this shows that any one of (i), (ii), (iii) is sufficient to insure that y is optimal.

To show necessity we assume that none of (i), (ii), (iii) holds and show that y fails to be optimal by inferring the existence of some $h \in S$ for which $\Phi(\epsilon, h) < 0$. In so doing we assume with no loss of generality that the interval T is open and that $\|\epsilon\| > 0$. Since (i) fails there exists some $\delta > 0$, some $s \in \{-1, +1\}$, and some interval partition $T = T_1 \cup \cdots \cup T_k$ of T with $1 \leqslant k \leqslant n$ (where $t_i < t_j$ whenever $t_i \in T_i$, $t_j \in T_j$, and $1 \leqslant i < j \leqslant k$) such that

$$\sup\{s \cdot (-1)^{i-1} \cdot \epsilon(t) : t \in T_i\} = \|\epsilon\|, \quad i = 1,..., k \tag{7}$$

$$\sup\{s \cdot (-1)^{i} \cdot \epsilon(t) : t \in T_i\} \leqslant \|\epsilon\| - \delta, \quad i = 1,..., k. \tag{8}$$

The constructions used to prove Theorems 4.2, 5.1, and 5.2 in [5] can be extended from the case where S is a Haar space on a compact interval to the present case where S is a Haar space of bounded continuous functions on the open interval T. This being the case there is some $h_0 \in S$ such that $\operatorname{sgn} h_0(t) = s \cdot (-1)^i$ whenever t is in the interior of T_i, $i = 1,..., k$ and such that

$$\sup\{h_0(t) \operatorname{sgn} \epsilon(t): t \in K \text{ and } |\epsilon(t)| = \|\epsilon\|\} < 0 \text{ whenever } K \subset T \text{ is compact.} \quad (9)$$

It follows that $\Phi(\epsilon, h_0) \leqslant 0$ with the inequality being strict so that y_0 fails to be optimal except in the case where $\limsup |\epsilon(t)| = \|\epsilon\|$ while $\lim h_0(t) = 0$ as t approaches some one of the endpoints of T.

To complete the proof we will show that in this exceptional case we may slightly perturb the above function h_0 to obtain some $h \in S$ for which $\Phi(\epsilon, h) < 0$. In the process we assume that limits involving the variables t_l, t_r are always taken as t_l, t_r approach the left, right endpoints of T, respectively.

Suppose first that this anomalous limiting behavior occurs only at one of the endpoints of T. For definiteness, we assume that $\limsup |\epsilon(t_l)| < \|\epsilon\|$ while $\limsup |\epsilon(t_r)| = \|\epsilon\|$ and $\lim h_0(t_r) = 0$. Since (ii) fails we can find $h_r \in S$ such that $\lim h_r(t_r) = s \cdot (-1)^k$ and then choose $\alpha_r > 0$ so small that (9) holds when we replace h_0 by $h = h_0 + \alpha_r h_r$. By construction $\Phi(\epsilon, h) < 0$. The same argument holds if we replace the assumption that $\limsup |\epsilon(t_l)| < \|\epsilon\|$ by the assumption that $\limsup |\epsilon(t_l)| = \|\epsilon\|$ provided that $\lim h_0(t_l) \neq 0$ so that we can also require α_r to be so small that $\alpha_r \lim |h_r(t_l) < \lim |h_0(t_l)|$.

Finally, we deal with the case where the anomalous limiting behavior simultaneously occurs at both endpoints so that

$$\limsup |\epsilon(t_l)| = \limsup |\epsilon(t_r)| = \|\epsilon\| \quad (10)$$

while $\lim h_0(t_l) = \lim h_0(t_r) = 0$. Since (ii) fails we can find h_l, $h_r \in S$ such that

$$\lim h_l(t_l) = -s, \lim h_r(t_r) = s \cdot (-1)^k. \quad (11)$$

If it is possible to choose such h_l, h_r so that the vectors

$$[\lim h_l(t_l), \lim h_l(t_r)], [\lim h_r(t_l), \lim h_r(t_r)] \quad (12)$$

are linearly independent, then (after replacing h_l, h_r by suitable linear combinations of h_l, h_r, if necessary) we can also arrange to have

$$\lim h_l(t_r) = \lim h_r(t_l) = 0$$

so that we again have $\Phi(\epsilon, h) < 0$ whenever $h = h_0 + \alpha_r h_r + \alpha_l h_l$ and α_l, $\alpha_r > 0$ are sufficiently small.

If it is impossible to choose h_l, h_r so that the vectors (12) are independent, then both $\lim h_r(t_l)$ and $\lim h_r(t_r)$ are nonzero with

$$\lim h_r(t_l) \cdot \lim h(t_r) = \lim h_r(t_r) \cdot \lim h(t_l) \qquad \text{whenever} \quad h \in S. \tag{13}$$

Moreover, we may assume that h_r has exactly $n - 1$ zeros in T so that

$$(-1)^{n-1} \lim h_r(t_l) \lim h_r(t_r) > 0. \tag{14}$$

Indeed, if this is not already the case we replace h_r with $\alpha(h_r + \beta h_z)$ where $h_z \in S$ has exactly $n - 1$ distinct zeros in T, where β is so large in magnitude that $h_r + \beta h_z$ also has $n - 1$ zeros in T and $\lim[h_r(t_r) + \beta h_z(t_r)] \neq 0$, and where the scale factor α is then adjusted so as to preserve the requirement that the new h_r satisfy (11). Together (13), (14) imply that

$$(-1)^n \lim h(t_l) \lim h(t_r) \leqslant 0 \qquad \text{whenever} \quad h \in S,$$

and since (10) holds and yet (iii) fails we have

$$\limsup\, (-1)^{n-1}\, \epsilon(t_l)\, \epsilon(t_r) = \| \epsilon \|^2. \tag{15}$$

From (7), (8), and (15) we conclude that $(-1)^k = (-1)^n$ so that

$$\begin{aligned} \operatorname{sgn} \lim h_r(t_l) = (-1)^{n-1} \lim \operatorname{sgn} h_r(t_r) &= (-1)^{k-1} \lim \operatorname{sgn} h_0(t_r) \\ &= \lim \operatorname{sgn} h_0(t_l) = -s. \end{aligned}$$

This being the case, if we set $h = h_0 + \alpha_r h_r$ we have $\Phi(\epsilon, h) < 0$ whenever $\alpha_r > 0$ is sufficiently small. ∎

The two examples given in the introduction serve to illustrate the various situations which are covered by conditions (i), (ii), and (iii) of the theorem. In addition, we present the following two corollaries (which may be proved by an immediate application of the above theorem.)

COROLLARY 1. *Let $f \in C_b[0, +\infty)$, let S be the n dimensional subspace of exponential sums given by* (2) *and* (3), *let $y \in S$, and let $\epsilon = f - y$.*

(a) *If $\lambda_n = 0$ (so that the constant functions are included in S) then y is a best uniform approximation to f on $[0, +\infty)$ if and only if ϵ essentially alternates at least n times on $[0, +\infty)$.*

(b) *If $\lambda_n < 0$ (so that every $h \in S$ vanishes at $+\infty$) then y is such a best approximation if and only if either ϵ essentially alternates at least n times on*

$[0, +\infty)$ or else $\limsup |f(t)| = \|\epsilon\|$ *as* $t \to +\infty$. *Moreover, if* $\lambda_n < 0$ *and* $\lim f(t) = 0$ *as* $t \to +\infty$ *then* y *is such a best approximation if and only if* ϵ *alternates at least* n *times on* $[0, +\infty)$.

Note. An extension of this result which applies in the case where the exponents λ_i are also allowed to vary is given in [4].

COROLLARY 2. *Let* $f \in C_b(-\infty, +\infty)$, *let* S *be the* n *dimensional subspace of functions of the form* $P(t) \cdot \exp(-t^2)$ *where* P *is a polynomial of degree* $n - 1$ *or less, let* $y \in S$, *and let* $\epsilon = f - y$. *Then* y *is a best uniform approximation to* f *on* $(-\infty, +\infty)$ *if and only if either* ϵ *essentially alternates at least* n *times on* $(-\infty, +\infty)$ *or* $\limsup |f(t)| = \|\epsilon\|$ *as* $t \to -\infty$ *or as* $t \to +\infty$. *If* $\lim f(t) = 0$ *as* $t \to \pm\infty$, *then* y *is such a best approximation if and only if* ϵ *alternates at least* n *times on* $(-\infty, +\infty)$.

3. CONSTRUCTION

When T is compact it is possible to use one of the Remez exchange algorithms (cf. [7, p. 105–116]) to numerically determine the unique best uniform approximation to a given $f \in C(T)$ on T from the n dimensional Haar subspace S of $C(T)$. In the sense made clear by the following theorem this (at least in principle) enables us to construct a best uniform approximation to f on T even when T is not compact.

THEOREM 3. *Let* S *be an* n *dimensional Haar subspace of* $C_b(T)$ *and let* $f \in C_b(T)$. *Let* $K_1 \subseteq K_2 \subseteq \cdots$ *be an expanding sequence of nondegenerate compact intervals with union* T, *and for each* $\nu = 1, 2, \ldots$ *let* y_ν *be the unique best uniform approximation to* f *on* K_ν. *Then some subsequence of* $\{y_\nu\}$ *converges uniformly on* T *to a best uniform approximation,* y, *to* f *on* T *from* S *with the corresponding error curve* $\epsilon = f - y$ *essentially alternating at least* n *times on* T.

Proof. The sequence $\{y_\nu\}$ is uniformly bounded on K_1 and since S is a finite dimensional subspace of $C_b(T)$ we may assume (after passing to a subsequence, if necessary) that $\{y_\nu\}$ uniformly converges on K_1 and thus on all of T to some $y \in S$. Given any $\delta > 0$ we can therefore find some index ν such that

$$\|y_\nu - y\| < \delta/3 \tag{16}$$

$$\|f - y\| < \|f - y\|_{K_\nu} + \delta/3 \tag{17}$$

where $\| \, \|_{K_\nu}$ denotes the sup seminorm on K_ν. Since y_ν is a best uniform approximation to f from S on K_ν, there exist points $t_0 < t_1 < \cdots < t_n$ from K_ν and $s \in \{-1, +1\}$ such that

$$s \cdot (-1)^i \cdot [f(t_i) - y_\nu(t_i)] = \| f - y_\nu \|_{K_\nu}, \qquad i = 0, 1, \ldots, n. \tag{18}$$

Using (16)–(18) we now have

$$\begin{aligned} s \cdot (-1)^i \, [f(t_i) - y(t_i)] &= \| f - y_\nu \|_{K_\nu} + s \cdot (-1)^i \cdot [y_\nu(t_i) - y(t_i)] \\ &\geqslant \| f - y \|_{K_\nu} - 2 \| y_\nu - y \| \\ &> \| f - y \| - \delta, \; i = 0, 1, \ldots, n, \end{aligned}$$

and since $\delta > 0$ is arbitrary the error function $\epsilon = f - y$ must essentially alternate at least n times on T. The argument used in the proof of Theorem 2 shows that this is sufficient to insure that y is a best uniform approximation to f on T from S. ∎

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